ARTICLE NO. 79-4116

Distributed Control of Spinning Flexible Spacecraft

L. Meirovitch,* H. F. VanLandingham,† and H. Öz‡
Virginia Polytechnic Institute and State University, Blacksburg, Va.

A unified treatment of modal control of spinning flexible spacecraft is presented. The equations of motion of the spacecraft are hybrid, i.e., they consist of ordinary differential equations for the rotational motion and partial differential equations for the elastic motion. Problems involving control of distributed-parameter systems are generally discretized and the actual control is implemented on the discrete systems. An efficient method for control of linear gyroscopic systems is via model synthesis. There remains, however, the question as to how modal control of discrete systems is related to the control of the actual distributed-parameter systems. It is the object of this paper to provide the information concerning the spatial distribution of the sensors and actuators and to relate this information to the discrete modal control.

I. Introduction

Owing to more complex mission requirements, there has been a trend in recent years toward larger and larger spacecraft. At the same time, there has been an increasing demand on precision attitude control of various spacecraft components, such as antennas. An obvious solution to the problem of providing the increasing power needed to control the spacecraft is to produce the power in space. This, in turn, requires relatively large solar panels whose orientation must also be controlled, although with less precision than the orientation of the antennas. In other cases, such as in the case of the solar power station satellite, the mission itself consists of collecting the solar energy.

Solar panels and some antennas tend to be highly flexible, so that not only the orientation needs to be controlled but also the elastic displacements. Unlike rigid rotors and platforms, elastic solar panels and antennas generally represent distributed parameter components, so that the dynamical system is really a hybrid one, i.e., a system described in terms of discrete and distributed coordinates. Such a system can be regarded as possessing an infinite number of degrees of freedom. Because there is no practical way of controlling large-order systems (and, in fact, it is not really necessary), truncation represents the only feasible alternative. The main problem is to develop a relatively low-order mathematical model without sacrificing essential dynamic characteristics of the system.

One commonly used discretization technique is lumping. In the case of large flexible structures, however, lumping is not an attractive discretization procedure, as it is likely to result in an unduly large-order mathematical model. Because a large flexible spacecraft consists of readily identifiable substructures, a variation on the component-mode synthesis appears as an ideally suited method. According to this method, the motion of every flexible substructure is represented as a superposition of a given number of "substructure modes" multiplied by time-dependent generalized coordinates. In essence, this method can be regarded as an extension of the assumed modes method[1] to structures consisting of an assemblage of substructures. This method has been used almost exclusively for nongyroscopic systems and is based on the stationarity of Rayleigh's quotient. It is shown in Ref. 2, however, that a stationarity principle exists also for gyroscopic systems, so that a discretization procedure based on the component-mode synthesis represents an attractive alternative for spacecraft with rotating components (and certainly for those with nonrotating components). But the stationarity principle of Ref. 2 goes considerably beyond the ideas underlying the basic method of component-mode synthesis. Indeed the component-mode synthesis advocates expansions in terms of substructure modes, whereas Ref. 2 shows that it is sufficient to use expansions in terms of admissible functions, provided that these functions belong to a complete set.

Studies of control problems for spacecraft with distributed elastic members have tended to treat lightly certain aspects connected with the distributed nature of sensing devices and actuators. For example, although Ref. 3 considers initially a hybrid mathematical model, the model is soon reduced to a discrete one. Then, the actual control is implemented on the discrete model via modal synthesis. The object of this paper is to provide the information concerning the spatial distribution of the sensors and actuators and to relate this information to the discrete modal control developed in Ref. 3.[§] The relation between the actual velocities of the distributed-parameter system and the generalized velocities of the discretized system as well as ways of determining the latter from sensor measurements are given in Secs. V and VI. This, in turn, enables one to define the requirements on the sensors distribution for the system to be observable. These relations for hybrid systems, particularly for distributed gyroscopic systems, appear to be derived for the first time in this paper. The relation between actual controls on the distributed-parameter system and "generalized controls" on the discretized system is vital to the understanding of control of distributed gyroscopic systems. The equations derived in this paper show clearly that a single actuator can reach all of the (discretized) generalized coordinates. Hence, for an n-degree-of-freedom system, the fact that the generalized control vector has full dimension should not be construed to imply a need for n actual actuators. Moreover, a given array of generalized controls can be realized in more than one way, as various combinations of actual controls can yield the same generalized control vector. The relation between actual

Presented at the First VPI&SU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Va., June 13-15, 1977; submitted Feb. 9, 1978; revision received Jan. 29, 1979.

Index categories: Spacecraft Dynamics and Control; Structural Dynamics; Analytical and Numerical Methods.

*Reynolds Metals Professor, Dept. of Engineering Science and Mechanics. Associate Fellow AIAA.
†Professor, Dept. of Electrical Engineering. Member AIAA.
‡Instructor, Dept. of Engineering Science and Mechanics.

§These topics are not discussed in Ref. 3, nor anywhere else.

controls and generalized controls for distributed gyroscopic systems also appears to be derived for the first time in this paper (Sec. VII).

The basic approach to the control problem used in this paper is *modal synthesis*, first presented in Ref. 3. It appears that the term modal control means different things to different people. For example, Simon and Mitter[4] refer to a scheme for the modification of the system eigenvalues as *modal control*. Reference 4 expresses the eigenvalues of the open-loop system in a Jordan form, but is not concerned with the decoupling of the closed-loop system. A similar control philosophy is proposed independently by Porter and Crossley,[5] although the computational details of Ref. 5 differ from those of Ref. 4. Once again, no system decoupling is sought. The procedures of Refs. 4 and 5 are generally known as *poles allocation* techniques. The approach of this paper is more in line with that described by Brogan.[6] The procedure of Ref. 6, referred to as *modal decomposition*, amounts simply to determining the Jordan form for the system, which requires the solution of the eigenvalue problem for the system, and effecting control in decoupled form. For general dynamical systems there are considerable computational difficulties in determining the system Jordan form. For linear gyroscopic systems, however, there are no particular computational problems.[7,8] Indeed, Jordan forms can be obtained for systems of order measuring in the hundreds or even thousands. The advantage of the decoupling procedure proposed in this paper is that it shifts the problem of high dimensionality from the control problem to the structural dynamics problem.[9] For high-order systems, decoupling may be the only feasible approach to control. It should be pointed out that many of the developments of this paper, although derived using a distributed gyroscopic system as a model, are applicable to distributed nongyroscopic systems as well.[10]

II. The Hybrid Equations of Motion

Let us consider a spacecraft consisting of some rigid parts occupying the domains D_R and some elastic parts occupying the domains D_E. We shall assume that there is a central rigid body and perhaps some other rigid bodies that do not change their position relative to the central rigid body, such as in the case of a symmetric gyro spinning about an axis fixed with respect to the central rigid body. For simplicity we shall assume that the mass center of the spacecraft in undeformed state coincides with the combined mass center of the rigid bodies, and hence with the combined mass center of the elastic bodies in undeformed state. These restrictions are not as severe as it may appear, although they imply certain inertial symmetry of the spacecraft. In addition, we shall ignore the motion of the mass center on the assumption that it is small.

It will prove convenient to work with a system of axes xyz attached to the central rigid body and describe the motion of the spacecraft in terms of the rotation of the frame xyz and the elastic deformtion of the elastic members relative to this frame (Fig. 1). We shall identify two major equilibrium states in which 1) the frame xyz is fixed in an inertial space but one rigid rotor spins with the uniform angular velocity Ω relative to xyz or 2) the frame xyz, and hence the entire spacecraft, rotates about a symmetry axis with the constant angular velocity Ω relative to the inertial space. In the latter case, it is possible for some elastic member to experience static deformations due to centrifugal forces, where the deformation pattern defines the so-called nontrivial equilibrium. In both cases, the motion can be described in terms of the angular perturbations θ_i $(i=1, 2, 3)$ of the axes xyz and the elastic oscillation $\boldsymbol{u}$ relative to xyz. It should be pointed out that the elastic oscillation does not include any possible constant deformation defining the nontrivial equilibrium, so that elastic oscillation can be small even in cases in which the elastic deformation is relatively large.

As long as the equilibrium state involves uniform rotation of one part, or of the entire spacecraft, the system is gyroscopic. For gyroscopic systems, the kinetic energy can be written in the general form

$$T=T_2(\theta_i,\dot{\theta}_i,\boldsymbol{u},\dot{\boldsymbol{u}})+T_1(\theta_i,\dot{\theta}_i,\boldsymbol{u},\dot{\boldsymbol{u}})+T_0(\theta_i,\boldsymbol{u}) \qquad (1)$$

where T_2 is quadratic in the velocities $\dot{\theta}_i$ and $\dot{\boldsymbol{u}}$, T_1 is linear in these velocities, and T_0 contains no velocities. The term T_1 leads to gyroscopic effects and T_0 to centrifugal effects in the system equations of motion.

The potential energy can arise from various sources, such as gravity and body elasticity. We shall assume, for simplicity, that the potential energy is due entirely to elastic effects and

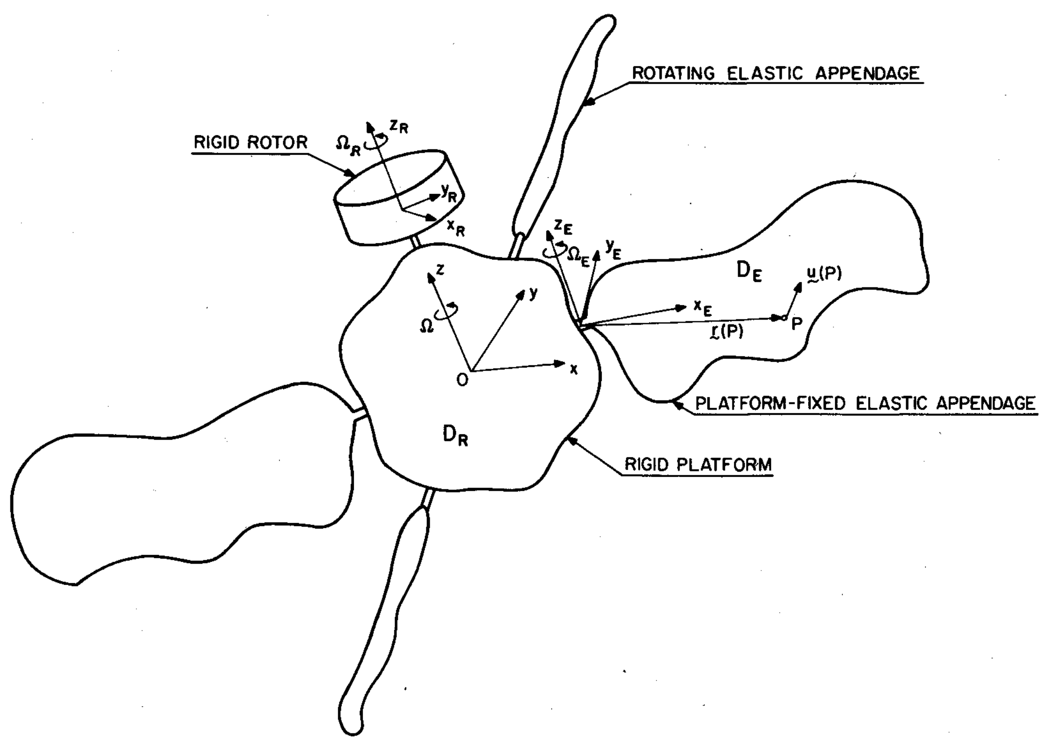

Fig. 1 General flexible spacecraft configuration.

can be written in the general form

$$V=(Au,u)=\int_{D_E} u^T A_D u \mathrm{d}D_E+\int_{S_E} u^T A_S u \mathrm{d}S_E \tag{2}$$

where A is a "two-sided" operator, containing partial derivatives with respect to the spatial variables. The operator A is assumed to be symmetric and positive definite and can be identified as an *energy operator.* Equation (2) represents a symbolic operation involving integrations over the elastic domains D_E and their boundaries S_E.

Denoting by R_R and R_E the absolute positions of points in the rigid and elastic parts of the spacecraft, the virtual work due to external and control forces can be written in the form

$$\delta W=\int_{D_R} f^T \delta R_R \mathrm{d}D_R+\int_{D_E} f^T \delta R_E \mathrm{d}D_E \tag{3}$$

where f is a distributed force vector and δR_R and δR_E are associated virtual displacements. These virtual displacements can be expressed in terms of the virtual rotation $\delta\theta$ and the virtual elastic displacement δu as follows:

$$\delta R_R=\tilde{r}^T\delta\theta \quad \delta R_E=[\tilde{r}+\tilde{u}]^T\delta\theta+\delta u\cong\tilde{r}^T\delta\theta+\delta u \tag{4}$$

where the product $\tilde{u}^T\delta\theta$ has been ignored as being of second order. The symbols $\tilde{r}$ and $\tilde{u}$ represent skew symmetric matrices associated with the nominal position vector $r=[x\ y\ z]^T$ and elastic displacement vector $u=[u_x\ u_y\ u_z]^T$. The matrix corresponding to r is

$$\tilde{r}=\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{5}$$

and a similar matrix exists for $\tilde{u}$, although it is no longer needed. Introducing Eqs. (4) into Eq. (3), we can write the virtual work,

$$\delta W\cong\int_{D_R} f^T\tilde{r}^T\delta\theta \mathrm{d}D_R+\int_{D_E} f^T(\tilde{r}^T\delta\theta+\delta u)\mathrm{d}D_E$$

$$=\left(\int_D f^T\tilde{r}^T \mathrm{d}D\right)\delta\theta+\int_{D_E} f^T\delta u \mathrm{d}D_E \tag{6}$$

where $D=D_R\cup D_E$ is the union of the two sets.

The equations of motion can be obtained by the extended Hamilton's principle[11]

$$\int_{t_1}^{t_2}(\delta L+\delta W)\mathrm{d}t=0 \tag{7a}$$

subject to

$$\delta\theta=\delta u=0 \quad \text{at } t=t_1,\ t_2 \tag{7b}$$

where $L=T-V$ is the system Lagrangian. The resulting Lagrange's equations are hybrid, in the sense that the equations for θ are ordinary differential equations and those for u are partial differential equations. Of course, the equations are simultaneous. For our purposes it will prove more convenient to work with a discrete system, instead of a hybrid one, so that in the next section we shall discretize the system.

III. Discretized Equations of Motion

Distributed-parameter systems are described in terms of coordinates depending on both space and time, so that a solution needs to be defined at every point P of the domain of extension of the distributed member. Because there are infinitely many such points, a distributed-parameter system can be regarded as possessing an infinity of degrees of freedom. Practical methods do not exist yet to control systems with an infinite number of degrees of freedom. In fact, the task of controlling a system with a large number of degrees of freedom is often sufficiently difficult to discourage any serious attempt. Hence, it is imperative that the distributed-parameter system be simulated by a discrete model, one which has as few degrees of freedom as possible and yet does not sacrifice essential dynamical features of the system. There are various discretization procedures in current use. We shall use a discretization method based on series expansions. Hence, let us represent the elastic displacement by

$$u(P,t)=\Phi(P)\zeta(t) \tag{8}$$

where $\Phi(P)$ is a $3\times m$ matrix of known functions of the spatial variables and $\zeta(t)$ is an m vector of time-dependent generalized coordinates. The dimension of the vector ζ represents the number of degrees of freedom used to simulate the distributed member.

By virtue of the assumption that the angular perturbations θ_i $(i=1, 2, 3)$ and elastic displacements u are small, the Lagrangian of the discretized system can be written in the quadratic form

$$L=\tfrac{1}{2}\dot\theta^T m_{\theta\theta}\dot\theta+\dot\theta^T m_{\theta\zeta}\dot\zeta+\tfrac{1}{2}\dot\zeta^T m_{\zeta\zeta}\dot\zeta+\theta^T f_{\theta\theta}\dot\theta+\theta^T f_{\theta\zeta}\dot\zeta$$
$$+\zeta^T f_{\zeta\theta}\dot\theta+\zeta^T f_{\zeta\zeta}\dot\zeta-\tfrac{1}{2}\theta^T k_{\theta\theta}\theta-\theta^T k_{\theta\zeta}\zeta-\tfrac{1}{2}\zeta^T k_{\zeta\zeta}\zeta \tag{9}$$

where

$$m_{\theta\theta}=\frac{\partial^2 T_2}{\partial\dot\theta_i\partial\dot\theta_j} \qquad f_{\theta\theta}=\frac{\partial^2 T_1}{\partial\theta_i\partial\dot\theta_j} \qquad k_{\theta\theta}=-\frac{\partial^2 T_0}{\partial\theta_i\partial\theta_j}$$
$$(i,j=1,2,3) \tag{10a}$$

$$m_{\theta\zeta}=\frac{\partial^2 T_2}{\partial\dot\theta_i\partial\dot\zeta_j} \quad f_{\theta\zeta}=\frac{\partial^2 T_1}{\partial\theta_i\partial\dot\zeta_j} \quad f_{\zeta\theta}=\frac{\partial^2 T_1}{\partial\dot\theta_i\partial\zeta_j} \quad k_{\theta\zeta}=-\frac{\partial^2 T_0}{\partial\theta_i\partial\zeta_j}$$
$$(i=1,2,3;\ j=1,2,\ldots,m) \tag{10b}$$

$$m_{\zeta\zeta}=\frac{\partial^2 T_2}{\partial\dot\zeta_i\partial\dot\zeta_j} \qquad f_{\zeta\zeta}=\frac{\partial^2 T_1}{\partial\zeta_i\partial\dot\zeta_j} \qquad k_{\zeta\zeta}=-\frac{\partial^2 T_0}{\partial\zeta_i\partial\zeta_j}+k_{E\zeta\zeta}$$
$$(i,j=1,2,\ldots,m) \tag{10c}$$

in which all of the derivatives are to be evaluated at equilibrium. Moreover,

$$k_{E\zeta\zeta}=\int_{D_E}\Phi^T A_D\Phi \mathrm{d}D_E+\int_{S_E}\Phi^T A_S\Phi \mathrm{d}S_E \tag{11}$$

and note that the integrals in Eq. (11) are to be interpreted in the same sense as those in Eq. (2). There remains the question as to the nature of the matrix Φ. An obvious approach is to choose the elements of Φ as the "modes" of the elastic member considered. Although this approach is perhaps the most appealing from the intuitive point of view, it is often difficult to implement. Moreover, the approach is not mandatory as one may be tempted to believe. The difficulty with using so-called "appendage modes" is that the eigenvalue problem associated with a given elastic member may not be readily defined, and, even if the eigenvalue problem can be defined, its solution may require substantial numerical work. Invoking a stationarity principle for gyroscopic systems,[2] however, appendage modes are not really necessary and any complete set of admissible functions should yield equally satisfactory results. Such admissible functions are sometimes called "energy functions" because they satisfy all of the conditions implied by the definition of the energy operator of Eq. (2).[12]

Introducing the n-dimensional configuration vector

$$q=[\theta^T \,|\, \zeta^T]^T \tag{12}$$

$$m=\begin{bmatrix} m_{\theta\theta} & m_{\theta\zeta} \\ m_{\theta\zeta}^T & m_{\zeta\zeta}\end{bmatrix} \quad f=\begin{bmatrix} f_{\theta\theta} & f_{\theta\zeta} \\ f_{\zeta\theta} & f_{\zeta\zeta}\end{bmatrix} \quad k=\begin{bmatrix} k_{\theta\theta} & k_{\theta\zeta} \\ k_{\theta\zeta}^T & k_{\zeta\zeta}\end{bmatrix} \tag{13}$$

where $n=3+m$, the Lagrangian can be written in the compact form

$$L=\tfrac{1}{2}\dot{q}^T m\dot{q}+q^T f\dot{q}-\tfrac{1}{2}q^T kq \tag{14}$$

Next, let us introduce Eq. (8) into Eq. (6) and write

$$\delta W=\left(\int_D f^T\tilde{r}^T \mathrm{d}D\right)\delta\theta+\left(\int_{D_E} f^T\Phi \mathrm{d}D_E\right)\delta\zeta \tag{15}$$

Then, recognizing that

$$[\delta\theta^T \,|\, \delta\zeta^T]^T=\delta q \tag{16}$$

represents the virtual displacement vector for the entire system, the vector

$$Q=\begin{bmatrix} \int_D \tilde{r}f\,\mathrm{d}D \\ \int_{D_E} \Phi^T f\,\mathrm{d}D_E \end{bmatrix} \tag{17}$$

can be identified as the associated generalized force vector. Introducing Eqs. (16) and (17) into Eq. (15), we obtain the virtual work in terms of generalized forces and generalized virtual displacements

$$\delta W=Q^T\delta q \tag{18}$$

which completes the discretization of the system.

Lagrange's equations for discrete systems can be written in the symbolic vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=Q \tag{19}$$

where L is given by Eq. (14) and Q by Eq. (17). Performing the indicated differentiations, we obtain the explicit form of Lagrange's equations

$$m\ddot{q}+g\dot{q}+kq=Q \tag{20}$$

where

$$g=f^T-f \tag{21}$$

is a skew symmetric "gyroscopic matrix."

Equation (20) represents a set of n second-order ordinary differential equations. The solution is obtained most conveniently by transforming the equations into a set of $2n$ first-order equations. To this end, introduce the $2n$-state vector and associated "force" vector

$$x=[\dot{q}^T \,|\, q^T]^T \qquad X=[Q^T \,|\, 0^T]^T \tag{22}$$

as well as the $2n\times 2n$ matrices

$$I=\begin{bmatrix} m & 0 \\ 0 & k\end{bmatrix} \qquad G=\begin{bmatrix} g & k \\ -k & 0\end{bmatrix} \tag{23}$$

where I is symmetric and G is skew symmetric, so that the desired set of $2n$ first-order equations has the form

$$I\dot{x}+Gx=X \tag{24}$$

A modal analysis for the solution of Eq. (24) has been presented in Refs. 7 and 8.

IV. Control by Modal Synthesis

The basic guidelines for modal control of discrete systems were presented in detail in Ref. 3. The general feedback form is shown in Fig. 2. From this diagram, it is clear that physical measurements taken on the spacecraft must be used to provide information to the observer subsystem. The system dynamics used in Ref. 3 is as given by Eq. (24), with the exception that here the vector X includes not only forces from external sources, such as meteorite impacts or solar radiation pressure, but also the generated control inputs.

Using the method presented in Ref. 8, the set of simultaneous equations, Eq. (24), can be reduced to a set of independent equations by means of the linear transformation

$$x=Pw \tag{25}$$

where P is the normalized modal matrix, $P^TIP=1$. The independent set can be written in the compact vector form

$$\dot{w}=Aw+P^TX \tag{26}$$

where

$$A=\text{block-diag}\begin{bmatrix} 0 & \omega_r \\ -\omega_r & 0\end{bmatrix} \quad (r=1,\ldots,n) \tag{27}$$

If one assumes that measurements vector z is related to the state vector x by $z=C_0x$, then the equivalent input to the decoupled observer can be written as

$$z=C_0Pw \tag{28}$$

However, it will prove particularly useful to work with the measurement rate

$$\dot{z}=C_0\dot{x} \tag{29}$$

where we note that the $\dot{x}$ vector now involves the system accelerations and velocities. Such a measurement vector is more consistent with the physical system because in actual

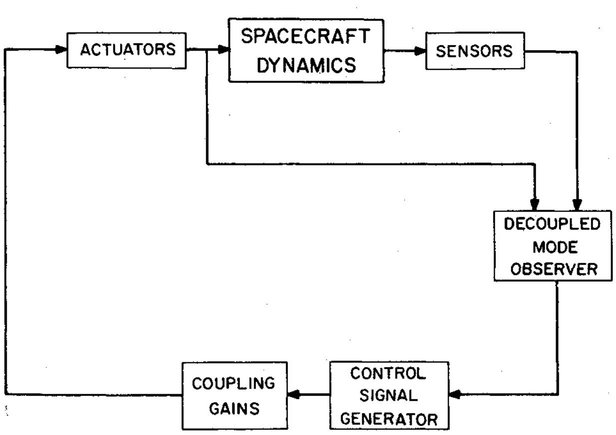

Fig. 2 General control system structure.

applications accelerations and velocities are the quantities measured. In terms of the decoupled state vector, Eq. (29) takes the form

$$\dot{z}=C_0P\dot{w} \tag{30}$$

Denoting the estimated decoupled state vector by $\hat{w}$, the observer dynamics can be written in the form

$$\dot{\hat{w}}=A_0\hat{w}+B_0\dot{z}+N_0P^TU \tag{31}$$

where U represents the control vector on the coupled system dynamics and the matrices A_0, B_0, and N_0 are chosen such that the error between the actual and the estimated states, i.e., $\epsilon(t)=w(t)-\hat{w}(t)$, decays with time. Introducing Eq. (30) into Eq. (31), the observer dynamics takes the form

$$\dot{\hat{w}}=A_0\hat{w}+B_0C_0P\dot{w}+N_0P^TU \tag{32}$$

After estimating the decoupled state vector w by $\hat{w}$, the control law can be formulated for each mode indpendently. This modal control law can be formulated in the form of proportional control or on-off, relay-type control. In this paper, we do not confine ourselves to any particular control law, so that the treatment is valid for both linear and nonlinear controls. If we denote the modal control vector on the decoupled dynamics by u^*, then the control vector U on the coupled system dynamics can be synthesized from u^* by means of modal synthesis. Note that in the case in which the components of u^* are of the on-off type, the components of U are said to be quantized, i.e., they have a staircase appearance. Because of the complexity involved in nonlinear control of high-order systems, the ability to decouple the system equations into a set of independent pairs of second-order system is an important asset in constructing quantized controls. The problem of control implementation is addressed in detail in Ref. 9. In principle, the control loop is now complete in that measurements from the spacecraft have been used to observe the behavior of the system and control law has been selected in a state feedback approach implementing the observer.

V. Generalized Velocities in Terms of Absolute Velocites

Let us consider the motion of an elastic member. In view of the discussion of Sec. II, the absolute position of a point mass P in the elastic member is

$$R_E(P,t)=r(P)+u(P,t) \tag{33}$$

where $r(P)=r_{OE}+r_E(P)$ is the nominal position of point P and $u(P,t)$ is the elastic displacement relative to axes $x_Ey_Ez_E$ (Fig. 3). Then, denoting by Ω_E the angular velocity vector of axes $x_Ey_Ez_E$ relative to the inertial space, the absolute velocity of point P is (see Ref. 3)

$$v_E=[\tilde{r}+\tilde{u}]^T\Omega_E+\dot{u}=[\tilde{r}+\tilde{u}]^TL_{PE}\Omega_P+\dot{u} \tag{34}$$

where L_{PE} is the matrix of direction cosines between xyz and $x_Ey_Ez_E$, Ω_P is the angular velocity vector of axes xyz, and $\dot{u}$ is the elastic velocity of P relative to the frame $x_Ey_Ez_E$.

Next, let us assume that in equilibrium the axes xyz rotate with the uniform angular velocity Ω about z when perturbed slightly. Denoting by $\dot{\theta}_1,\dot{\theta}_2,\dot{\theta}_3$ the perturbed angular velocities about x,y,z, respectively, and letting the perturbed angles $\theta_1,\theta_2,\theta_3$ be small, the linearized angular velocity vector becomes (see Ref. 3)

$$\Omega_P=[-\Omega\theta_2+\dot{\theta}_1\ \ \Omega\theta_1+\dot{\theta}_2\ \ \Omega+\dot{\theta}_3]^T \tag{35}$$

For convenience, let us assume that the elastic member lies in the xy plane and that the elastic displacement component u_x is relatively small, so that

$$R_E=[x\ \ y+u_y\ \ u_z]^T \tag{36}$$

Moreover, assuming that axes $x_Ey_Ez_E$ are parallel to axes xyz, so that L_{PE} reduces to the identity matrix, and ignoring second-order terms, we can insert Eqs. (35) and (36) in Eq. (34) and obtain

$$v_E\cong\begin{bmatrix}-(\Omega y+\dot{\theta}_3y+\Omega u_y)\\(\Omega+\dot{\theta}_3)x+\dot{u}_y\\(-\Omega\theta_2+\dot{\theta}_1)y-(\Omega\theta_1+\dot{\theta}_2)x+\dot{u}_z\end{bmatrix} \tag{37}$$

The coordinate θ_3 can be shown to be ignorable. Indeed, from Ref. 3, we can write

$$C\dot{\theta}_3+\int_m x\dot{u}_y\mathrm{d}m=0 \tag{38}$$

where C is the moment of inertia of the entire spacecraft about axis z. Introducing Eq. (38) into Eq. (37) and retaining only the y and z components, we obtain

$$v_y=\left(\Omega-\frac{1}{C}\int_m x\dot{u}_y\mathrm{d}m\right)x+\dot{u}_y \tag{39a}$$

$$v_z=(\dot{\theta}_1-\Omega\theta_2)y-(\dot{\theta}_2+\Omega\theta_1)x+\dot{u}_z \tag{39b}$$

Next, let us recall the discretization scheme, Eq. (8), and denote the second and third rows of the matrix Φ by Φ_y^T and Φ_z^T, respectively, so that

$$u_y=\Phi_y^T\zeta \qquad u_z=\Phi_z^T\zeta \tag{40}$$

Introducing Eqs. (40) into Eqs. (39), we can write

$$v_y-\Omega x=\left(\frac{x}{C}a^T+\Phi_y^T\right)\dot{\zeta} \tag{41a}$$

$$v_z-(\dot{\theta}_1-\Omega\theta_2)y+(\dot{\theta}_2+\Omega\theta_1)x=\Phi_z^T\dot{\zeta} \tag{41b}$$

where

$$a=-\int_m x\Phi_y\mathrm{d}m \tag{42}$$

Equations (41) represent the relation between the absolute velocities defining the motion of the central body and that of every point in the elastic member and the generalized velocity vector $\dot{\zeta}$.

Equations (41) contain m generalized velocities $\dot{\zeta}_r(t)$. In addition, they contain the angles $\theta_1(t)$ and $\theta_2(t)$ and their time derivatives. The angular motion can be measured directly by means of rate gyros mounted on the rigid platform. To get an estimate of the generalized coordinates ζ_r, we need some observations of the motion of the elastic member. The number of observations depends on the integer m. Of course, at least in theory, if the elastic member is regarded as continuous, then an exact estimation of the motion of every point P requires an infinite number of observations, i.e., conceptually $m\to\infty$. In practice, we must choose m as finite.

VI. Determination of Generalized Velocities

Let us assume that we wish to determine $\dot{\zeta}_r(t)$ $(r=1,2,...,m)$ by means of k observations of v_y and $m-k$ observations of v_z. These observations can be made by means of accelerometers, which sense absolute accelerations, in conjunction with an integration with respect to time to obtain velocities. We shall denote the points at which the observations are made by P_i $(1 \le i \le k)$ and $P_j (k+1 \le j \le m)$, respectively, and we note that the points P_i need not all be different from the points P_j. With this notation, Eqs. (41) yield

$$v_y(P_i,t) - \Omega x_{Pi} = \left[\frac{x_{Pi}}{C} a^T + \Phi_y^T(P_i)\right]\dot{\zeta} \quad (1 \le i \le k) \tag{43a}$$

$$v_z(P_j,t) - (\dot{\theta}_1 - \Omega\theta_2) y_{Pj} + (\dot{\theta}_2 + \Omega\theta_1) x_{Pj} = \Phi_z^T(P_j)\dot{\zeta} \quad (k+1 \le j \le m) \tag{43b}$$

where x_{Pi} represents the value of x corresponding to the point P_i, etc. It will prove convenient to introduce the m vector of measurements

$$v = [v_y^T \mid v_z^T]^T \tag{44}$$

where

$$v_y = \begin{bmatrix} v_y(P_1,t) - \Omega x_{P1} \\ v_y(P_2,t) - \Omega x_{P2} \\ \hdashline \\ v_y(P_k,t) - \Omega x_{Pk} \end{bmatrix} \tag{45a}$$

$$v_z = \begin{bmatrix} v_z(P_{k+1},t) - (\dot{\theta}_1 - \Omega\theta_2) y_{Pk+1} + (\dot{\theta}_2 + \Omega\theta_1) x_{Pk+1} \\ v_z(P_{k+2},t) - (\dot{\theta}_1 - \Omega\theta_2) y_{Pk+2} + (\dot{\theta}_2 + \Omega\theta_1) x_{Pk+2} \\ \hdashline \\ v_z(P_m,t) - (\dot{\theta}_1 - \Omega\theta_2) y_{Pm} + (\dot{\theta}_2 + \Omega\theta_1) x_{Pm} \end{bmatrix} \tag{45b}$$

Although the vectors Φ_y and Φ_z have dimension m, the last $m-k$ components of Φ_y and the first k components of Φ_z are generally zero. It will prove convenient to define the k vector Φ_y^* and $(m-k)$-vector Φ_z^*, obtained from Φ_y and Φ_z, respectively, by deleting the null elements just mentioned. Then, introducing the $m \times m$ matrix

$$\Phi^* = \begin{bmatrix} \Phi_y^* & 0 \\ 0 & \Phi_z^* \end{bmatrix} \tag{46}$$

where

$$\Phi_y^* = [\phi_{yir}^*] = [(x_{Pi}/C)a_r + \phi_{yr}^*(P_i)] \quad (i,r = 1,2,\ldots,k) \tag{47a}$$

is a square matrix of order k and

$$\Phi_z^* = [\phi_{zjr}^*] = [\phi_{zr}^*(P_j)] \quad (j,r = k+1,\ k+2,\ldots,m) \tag{47b}$$

is a square matrix of order $m-k$, Eqs. (43) can be written in the compact form

$$v = \Phi^* \dot{\zeta} \tag{48}$$

It follows from Eq. (48) that the vector $\dot{\zeta}$ of the generalized velocities can be inferred from observations by means of the formula

$$\dot{\zeta} = \Phi^{*-1} v = \begin{bmatrix} \Phi_y^{*-1} v_y \\ \hdashline \Phi_z^{*-1} v_z \end{bmatrix} \tag{49}$$

which requires, of course, that Φ_y^* and Φ_z^* be nonsingular. Hence, the sensors must be so located as not to produce matrices Φ_y^* and Φ_z^* with null columns. In particular, if the sensors measuring transverse motions lie on a nodal line of one of the functions Φ_{zr}^* $(k+1 \le r \le m)$, then it is clear from Eq. (47b) that the matrix Φ_z^* will have one null column. Nor should the sensors be clustered around such a nodal line, because this may cause the matrix Φ to become ill-conditioned. The vector v is generally obtained from measurements, a process which can lead to errors. This problem is the subject of a different investigation and we shall assume that the measurements provide exact values for the components of v.

As an example, let us consider one term for u_y and four terms for u_z, $k=1$, $m=5$. From Fig. 3, we see that the points are $P_1(L,0)$, $P_2(L,b)$, $P_3(L,-b)$, $P_4(l,b)$, and $P_5(l,-b)$. Then, the vector v becomes

$$v = \begin{bmatrix} v_y(L,0,t) - \Omega(r_{OE}+L) \\ v_z(L,b,t) - (\dot{\theta}_1 - \Omega\theta_2)b + (\dot{\theta}_2 + \Omega\theta_1)(r_{OE}+L) \\ v_z(L,-b,t) + (\dot{\theta}_1 - \Omega\theta_2)b + (\dot{\theta}_2 + \Omega\theta_1)(r_{OE}+L) \\ v_z(l,b,t) - (\dot{\theta}_1 - \Omega\theta_2)b + (\dot{\theta}_2 + \Omega\theta_1)(r_{OE}+l) \\ v_z(l,-b,t) + (\dot{\theta}_1 - \Omega\theta_2)b + (\dot{\theta}_2 + \Omega\theta_1)(r_{OE}+l) \end{bmatrix} \tag{50}$$

and the matrices Φ_y^* and Φ_z^* are

$$\Phi_y^* = [(r_{OE}+L)a_1/C + \Phi_{y1}^*(L,0)] \tag{51a}$$

$$\Phi_z^* = \begin{bmatrix} \phi_{z2}^*(L,b) & \phi_{z3}^*(L,b) & \phi_{z4}^*(L,b) & \phi_{z5}^*(L,b) \\ \phi_{z2}^*(L,-b) & \phi_{z3}^*(L,-b) & \phi_{z4}^*(L,-b) & \phi_{z5}^*(L,-b) \\ \phi_{z2}^*(l,b) & \phi_{z3}^*(l,b) & \phi_{z4}^*(l,b) & \phi_{z5}^*(l,b) \\ \phi_{z2}^*(l,-b) & \phi_{z3}^*(l,-b) & \phi_{z4}^*(l,-b) & \phi_{z5}^*(l,-b) \end{bmatrix} \tag{51b}$$

where Φ_y^* is really a scalar and not a matrix. Now, points P_4 and P_5 may lie on a nodal line of Φ_{z4}^* and Φ_{z5}^*, but P_2 and P_3 will not lie on the same nodal line. However, if P_4 and P_5 lie on a nodal line of Φ_{z4}^*, then $\Phi_{z4}^*(P_4) = \Phi_{z4}^*(P_5) = 0$, so that one diagonal element of Φ_z^* is zero. Whereas this may not render the matrix Φ^* singular, it may create problems in inverting Φ^* if the inversion algorithm requires that all of the diagonal elements of the matrix be nonzero. In general, the location of the sensors should be selected such that the matrix Φ^* be well-conditioned.

In the foregoing, the relation between measurements physically made on the spacecraft and the generalized velocities used on the dynamical modeling of the spacecraft has been developed. In particular we measured the angular velocities and the absolute velocities at the sensor locations. The sensor readings were converted to the generalized velocities by means of Eq. (49). Hence, as measurements we can consider the angular velocities and the generalized elastic

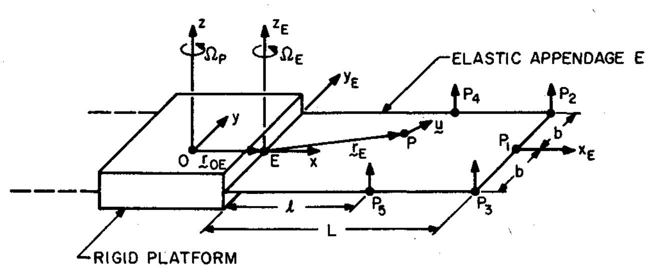

Fig. 3 Elastic appendage with distributed-sensor locations.

velocities, $\dot{\theta}_1, \dot{\theta}_2, \dot{\zeta}^T$. Let us denote the vector of measurements by $\dot{z}$ and conveniently call its components rate measurements because we are measuring time derivatives of the system coordinates. Recalling the form of the state vector $x^T = [\dot{\theta}_1 \; \dot{\theta}_2 \; \dot{\zeta}^T \; \theta_1 \; \theta_2 \; \zeta^T]$, we can write

$$\dot{z} = [I \mid 0]x \tag{52}$$

or, alternatively,

$$\dot{z} = [0 \mid I]\dot{x} \tag{53}$$

Comparing Eqs. (29) and (53), we conclude that

$$C_0 = [0 \mid I] \tag{54}$$

In practice, the rate measurement vector $\dot{z}$, which is used as an input to the observer given by Eq. (31) to estimate the modal state vector w, will be inferred from the gyro and sensor readings by means of the matrix transformation

$$\dot{z} = [\dot{\theta}_1, \dot{\theta}_2 \mid \dot{\zeta}^T]^T = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline 0 & & \Phi^{*-1} \end{array}\right] \left[\begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \hline v \end{array}\right] \tag{55}$$

where the transformation matrix is a square matrix of order $2+m$. By taking sufficient measurements to generate the generalized velocities, it is clear that the system is observable for the number of elastic modes considered. This, in turn, implies the existence of suitable matrices for designing our observer to function over all state variables.

Having conceptually tied the measurements on the spacecraft into the observer subsystem, and assuming that suitable actuator signals are constructed for feedback control, there remains to investigate the effect of physically located thrusters and torquing devices.

VII. Spatial Distribution of Actuators

The calculation of the generalized forces due to the actuators follows the pattern established in Sec. II. To this end, let us consider a given number of control torques $M_i \, (i=1,2,\ldots,k)$ and control forces $F_j \, (j=1,2,\ldots,l)$ to be concentrated throughout the spacecraft structure at points $P=P_i$ and $P=P_j$, respectively. These torques and forces can be represented as distributed torques and forces by writing them in the form $M_i\delta(P-P_i)$ and $F_j\delta(P-P_j)$, where $\delta(P-P_i)$ and $\delta(P-P_j)$ are spatial delta functions. We shall assume that the torques operate on the rigid platform and the forces on the elastic panels. The torques are designed to control the rigid body rotation and the forces the elastic motion.

Next, let us recall the second of Eqs. (4) for the virtual displacement vector δR_E of an arbitrary point of the elastic member, and write the virtual work due to control torques and forces in the form

$$\begin{aligned} \delta W_c &= \int_{S_R} M_i^T \delta(P-P_i)\delta\theta \mathrm{d}S_R + \int_{S_E} F_j^T \delta(P-P_j)\delta R_E \mathrm{d}S_E \\ &= \sum_{i=1}^{k} M_i^T(P_i,t)\delta\theta(t) + \sum_{j=1}^{l} F_j^T(P_j,t)[\tilde{r}(P_j) \\ &\quad + \tilde{u}(P_j,t)]^T\delta\theta(t) + \sum_{j=1}^{l} F_j^T(P_j,t)\delta u(P_j,t) \\ &\cong \left[\sum_{i=1}^{k} M_i^T(P_i,t) + \sum_{j=1}^{l} F_j^T(P_j,t)\tilde{r}^T(P_j)\right]\delta\theta \\ &\quad + \sum_{j=1}^{l} F_j^T(P_j,t)\delta u(P_j,t) \end{aligned} \tag{56}$$

where again the product $\tilde{u}^T(P_j,t)\delta\theta(t)$ is assumed to be of second order.

We shall assume that the torque vector has no component in the z direction and that the force vector has no component in the x direction. Hence,

$$M_i = \begin{bmatrix} M_{xi} \\ M_{yi} \\ 0 \end{bmatrix} \qquad F_j = \begin{bmatrix} 0 \\ F_{yj} \\ F_{zj} \end{bmatrix} \tag{57}$$

Moreover, from Eq. (5), we conclude that

$$\tilde{r}^T(P_j) = \begin{bmatrix} 0 & 0 & -y_{Pj} \\ 0 & 0 & x_{Pj} \\ y_{Pj} & -x_{Pj} & 0 \end{bmatrix} \tag{58}$$

Introducing Eqs. (57) and (58) into Eq. (56), we obtain

$$\begin{aligned} \delta W_c &= \left[\sum_{i=1}^{k} M_{xi}(P_i,t) + \sum_{j=1}^{l} y_{Pj}F_{zj}(P_j,t)\right]\delta\theta_1 \\ &+ \left[\sum_{i=1}^{k} M_{yi}(P_i,t) - \sum_{j=1}^{l} x_{Pj}F_{zj}(P_j,t)\right]\delta\theta_2 \\ &+ \sum_{j=1}^{l} x_{Pj}F_{yj}(P_j,t)\delta\theta_3 + \sum_{j=1}^{l} F_{yj}(P_j,t)\delta u_y(P_j,t) \\ &+ \sum_{j=1}^{l} F_{zj}(P_j,t)\delta u_z(P_j,t) \end{aligned} \tag{59}$$

Next, let us introduce the discretization scheme, Eq. (40), and write

$$\delta u_y = \Phi_y^T \delta\zeta \qquad \delta u_z = \Phi_z^T \delta\zeta \tag{60}$$

so that Eq. (59) becomes

$$\begin{aligned} \delta W_c &= \left[\sum_{i=1}^{k} M_{xi}(P_i,t) + \sum_{j=1}^{l} y_{Pj}F_{zj}(P_j,t)\right]\delta\theta_1 \\ &+ \left[\sum_{i=1}^{k} M_{yi}(P_i,t) - \sum_{j=1}^{l} x_{Pj}F_{zj}(P_j,t)\right]\delta\theta_2 \\ &+ \sum_{j=1}^{l} x_{Pj}F_{yj}(P_j,t)\delta\theta_3 + \sum_{j=1}^{l} [F_{yj}(P_j,t)\Phi_y^T(P_j) \\ &+ F_{zj}(P_j,t)\Phi_z^T(P_j)]\delta\zeta \end{aligned} \tag{61}$$

But θ_3 is ignorable, so that by analogy with Eq. (38), $\delta\theta_3$ satisfies

$$\delta\theta_3 = -\frac{1}{C}\int_m x\delta u_y \mathrm{d}m = \frac{1}{C}a^T\delta\zeta \tag{62}$$

where a is given by Eq. (42). Introducing Eq. (62) into Eq. (61), we can write

$$\delta W_c = \Theta_1\delta\theta_1 + \Theta_2\delta\theta_2 + \Xi^T\delta\zeta \tag{63}$$

where

$$\Theta_1 = \sum_{i=1}^{k} M_{xi}(P_i,t) + \sum_{j=1}^{l} y_{Pj}F_{zj}(P_j,t) \tag{64a}$$

$$\Theta_2 = \sum_{i=1}^{k} M_{yi}(P_i,t) - \sum_{j=1}^{l} x_{Pj}F_{zj}(P_j,t) \tag{64b}$$

$$\Xi^T = \sum_{j=1}^{l} \left[\frac{x_{pj}}{C} F_{yj}(P_j,t) a^T + F_{yj}(P_j,t) \Phi_y^T(P_j) + F_{zj}(P_j,t) \Phi_z^T(P_j) \right] \tag{64c}$$

are generalized forces. In terms of the spacecraft dynamics of Eq. (24), the forcing vector $X(t)$ consists of a desirable control term and disturbance terms. This control vector is $U(t) = [F^T \; \mathbf{0}^T]$, where $F^T = [\Theta_1 \; \Theta_2 \; \Xi^T]$ is obtained from Eqs. (64).

Equations (64) relate the generalized discretized control vector U to the actual control forces and torques. They imply that there is no one-to-one correspondence between the dimension of the vector U and the number of the physical actuators. They also imply that there is some arbitrariness in choosing the locations of the actuators, so that the vector U can be realized in many different ways. An example of control of a distributed gyroscopic system using the results obtained in this section is presented in Ref. 9.

VIII. Observer-Based Control

In the closed-loop control, the actuator commands are derived from the control signals which are, in turn, generated from observations of the decoupled coordinates as indicated in Fig. 2. In this section, we present the design of a stochastic observer, i.e., an optimal linear estimator which uses the measurements from the spatially distributed sensors. The use of decoupled dynamics is essential to the subsequent design of the relay-type controls since the design of a nonlinear controller for a high-order system is extremely intractable.

From Eq. (25) and the orthogonality relation $P^T I P = 1$, the decoupled state vector can be written as

$$w = P^T I x \tag{65}$$

The decoupled dynamic system can be represented by

$$\dot{w} = A w + B U + n \tag{66a}$$

$$\dot{z} = C \dot{w} + v \tag{66b}$$

where $A = -P^T G P$, $B = P^T$, and $C = C_0 P$. Again, U is the vector of deterministic control inputs to spacecraft and $\dot{z}$ is the measurement vector based on acceleration and velocity measurements. The vectors n and v are stochastic processes representing external disturbances on the spacecraft dynamics and measurement errors, respectively. In the following we assume that both n and v are zero-mean, white Gaussian, statistically independent random processes.

The objective of an observer is to estimate the state of the system as closely as possible, even though the measurements may be incomplete and influenced by noise. In the absence of stochastic disturbances and mesaurement errors, one has a deterministic observer. Before presenting a discussion of the observer dynamics, we will establish the conditions under which the system described by

$$\dot{w} = A w + B U \tag{67}$$

$$\dot{z} = C \dot{w} = C_0 P \dot{w} \tag{68}$$

is observable. We will address this question within the context of reconstructibility which is complementary to observability. To this end, we invoke the definition of reconstructibility as given by the theorem (Ref. 13, Sec. 1.7.1):

> "The state vector of the system described by Eq. (67) is completely reconstructible if and only if for all t_1, there exists a t_0 with $-\infty < t_0 < t_1$ such that the equality
>
> $$\dot{z}(t, t_0, w_0, U = \mathbf{0}) = \mathbf{0} \quad (t_0 \le t < t_1) \tag{69}$$
>
> implies that $w(t_0) = \dot{w}(t_0) = \mathbf{0}$."

In view of this theorem it is not difficult to show that the decoupled dynamics is reconstructible if the measurement matrix $C_0 P$ has independent rows, i.e., if the rank of $C_0 P$ is equal to its minimum dimension. With this condition satisfied, employing the definition of reconstructibility and the solution of the decoupled dynamics for zero input, and keeping in mind the special form of the A matrix, it can be shown that the rows of the observability matrix

$$\Psi = [(C_0P)^T \mid A^T (C_0P)^T \mid A^{2^T} (C_0P)^T \mid \ldots \mid A^{2n-1^T} (C_0P)^T]^T \tag{70}$$

span the $2n$ space. Hence, for the decoupled dynamics, instead of checking the rank of the observability matrix, it is only sufficient for the measurement matrix $C_0 P$ to have independent rows for the system to be observable.

Assuming that the system is observable and that the measurements are complete, the general linear system

$$\dot{\hat{w}} = A_0 \hat{w} + B_0 \dot{z} + N_0 P^T U \tag{71}$$

will establish the deterministic observer for the rate measurements. The stochastic processes n and v will be neglected temporarily, while we specify the observer matrices.

Because the observation error $\epsilon = w - \hat{w}$ is required to approach zero with time, we formulate the error dynamics in the form

$$\dot{\epsilon}(t) = \dot{w} - \dot{\hat{w}} = A w + P^T U - (A_0 \hat{w} + B_0 \dot{z} + N_0 P^T U) \tag{72}$$

Neglecting n and v in Eqs. (66), we substitute for $\dot{z}$ its equivalent expression in terms of w and U. After collecting like terms, we obtain

$$\dot{\epsilon} = (1 - B_0 C) A w - A_0 \hat{w} + (1 - N_0 - B_0 C) P^T U \tag{73}$$

Letting the matrices A_0, B_0, and N_0 satisfy the equations

$$(1 - B_0 C) A = A_0 \qquad 1 - B_0 C = N_0 \tag{74}$$

the equation for error dynamics becomes

$$\dot{\epsilon}(t) = A_0 \epsilon(t) \tag{75}$$

Hence, if the eigenvalues of A_0 have negative real parts, the error $\epsilon(t)$ will approach zero asymptotically, the rate of decay depending on the magnitude of these real parts. With minor reconstructibility conditions on the system structure, B_0 may be used to adjust the set of eigenvalues of A_0 arbitrarily, so that the observer response can be made arbitrarily fast. However, if noise is present in the system, then there will be restrictions on the choice of the observer eigenvalues. Indeed, the larger the negative real parts of the eigenvalues are, the more sensitive the observer will be to measurement errors. In this case the observer should respond slowly enough (which means smaller real parts for the eigenvalues) to provide some filtering of the data. Therefore, its design is a compromise between the speed of reconstruction and insensitivity to noise.

Introducing Eqs. (74) into Eq. (71), the observer dynamics becomes

$$\dot{\hat{w}} = A \hat{w} + P^T U + B_0 (\dot{z} - C A \hat{w} - C P^T U) \tag{76}$$

where B_0 is the observer gain matrix. If the rate measurement vector $\dot{z}$ has dimension m, where $m \leq 2n$, then C and B_0 become $m \times 2n$ and $2n \times m$ matrices, respectively. Introducing the matrix

$$B_0^* = B_0 C = B_0 C_0 P \tag{77}$$

we note that B_0^* is always a $2n \times 2n$ matrix, regardless of the dimension of the measurement vector $\dot{z}$. Using Eq. (77), the first of Eqs. (74) can be rewritten as

$$(I - B_0^*) A = A_0 \tag{78}$$

The actual gain matrix B_0 can be obtained from Eq. (77) as soon as the matrix B_0^* is determined. Indeed, if the system is observable and the measurement vector is complete, then the matrix C is invertible and we can write

$$B_0 = B_0^* C^{-1} = B_0^* (C_0 P)^{-1} \tag{79}$$

If the measurements are incomplete, then B_0 can be obtained from

$$B_0 = B_0^* (C_0 P)^T (C_0 P P^T C_0^T)^{-1} \tag{80}$$

The question remains whether $C_0 P P^T C_0^T$ is invertible. However, this is guaranteed by the fact that the system is observable. This, in turn, implies that $C_0 P$ and $C_0 P P^T C_0^T$ have the same rank.

The form of the observer gain matrix B_0 obtained above is not optimal in a noise environment indicated by Eqs. (66). Therefore, we will be more explicit concerning the gains B_0 in the presence of stochastic disturbances and measurement errors. The linear observer dynamics given by Eq. (76) can still be used. In this case, the observer is a Kalman stochastic estimator. If the noise processes are Gaussian the linear dynamics of Eq. (76) is the best estimator. However, if the system is influenced by non-Gaussian processes, then Eq. (76) describes the best linear estimator, all in a mean-square sense.

Let us assume, for simplicity, that both $\boldsymbol{n}$ and $\boldsymbol{v}$ are zero-mean, white noise, Gaussian random processes with intensities N and V, respectively, and each component of the measurement is influenced by noise. Then the gain matrix for the observer is given in the following matrix Riccati equation (see, for example, Ref. 13, Sec. 4.3.2)

$$\dot{M}(t) = AM(t) + M(t)A^T + N - M(t)C^T V^{-1} CM(t)$$

$$M(t_0) = M_0 \tag{81}$$

$$B_0(t) = M(t) A^T C^T V^{-1} \tag{82}$$

where $M(t)$ is the covariance matrix of the error between the actual state vector $\boldsymbol{w}(t)$ and the estimate $\hat{\boldsymbol{w}}(t)$ and M_0 is the covariance of the initial error. The stochastic observer dynamics given by these equations is optimal in the sense that it minimizes the mean square error. It is not possible to reduce the error to zero, because the system dynamics and the measurements are continuously contaminated by noise. Since we are dealing with a stochastic system, we can only speak of the mean of the estimation error, denoted by $\bar{\epsilon}(t)$. The error dynamics for the stochastic system can be shown to have the same form as that of Eq. (75)

$$\dot{\bar{\epsilon}}(t) = (I - B_0 C) A \bar{\epsilon}(t) \tag{83}$$

We can expect to make the mean error zero by choosing the initial condition of the observer as

$$\hat{\boldsymbol{w}}(t_0) = \bar{\boldsymbol{w}}(t_0) \tag{84}$$

where again an overbar denotes the mean. This is equivalent to making the initial mean error zero. Hence, Eq. (83) indicates that the mean error will remain zero for $t > t_0$.

As indicated by Eqs. (81) and (82), the optimal observer requires a time-varying gain matrix $B_0(t)$. However, $B_0(t)$ is based on the solution $M(t)$ of the matrix Riccati equation, Eq. (81). The actual gain matrix can be calculated in advance as the solution of this equation and stored with the controller. However, a more common procedure is to implement a constant gain matrix. For this purpose one can either use the steady-state solution of Eq. (81) or an empirically established gain matrix for acceptable performance based on simulation of Eq. (76). The latter approach becomes more attractive when there is less a priori knowledge concerning the system and measurement noise levels. One final remark is in order at this point. The expression for the gain matrix given by Eq. (82) requires that V be invertible. However, this is guaranteed because we assumed that each component of the measurement is influenced by noise, hence V is a full diagonal matrix.

After the design of an observer, the necessary signals for actuating the controls are available, namely, a knowledge of the components of $\hat{\boldsymbol{w}}(t)$. A detailed exposition of both proportional on-off control laws is presented in Ref. 3.

IX. Conclusions

Control of a spinning flexible spacecraft by modal synthesis is presented, taking into account the distributed nature of the system. Control laws based on the discretized system equations are related to the actuators, where the latter are distributed throughout the spacecraft. In addition, the relations between the generalized velocities and the absolute velocities are developed in terms of the physical location of the distributed sensors. Requirements for the reconstructibility of the state vector are given on the basis of the rate measurements and decoupled system dynamics. Design of a Luenberger-type observer is presented. In the case of stochastic processes, the discussion is extended to include the algorithm for a Kalman filter based on the decoupled dynamics.

References

[1] Meirovitch, L., *Analytical Methods in Vibrations*, The Macmillan Co., New York, 1967.

[2] Meirovitch, L., "A Stationarity Principle for the Eigenvalue Problem for Rotating Structures," *AIAA Journal*, Vol. 14, Oct. 1976, pp. 1387-1394.

[3] Meirovitch, L., VanLandingham, H. F., and Öz, H., "Control of Spinning Flexible Spacecraft by Modal Synthesis," *XXVII Congress, International Astronautical Federation*, Paper IAF-76-022, Oct. 1976; also *Acta Astronautica*, Vol. 4, 1977, pp. 985-1010.

[4] Simon, J. D. and Mitter, S. K., "A Theory of Modal Control," *Information and Control*, Vol. 13, 1968, pp. 316-353.

[5] Porter, B. and Crossley, R., *Modal Control, Theory and Applications*, Taylor and Francis Ltd., London, 1972.

[6] Brogan, W. L., *Modern Control Theory*, Quantum Publishers, Inc., New York, 1974.

[7] Meirovitch, L., "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal*, Vol. 12, Oct. 1974, pp. 1337-1342.

[8] Meirovitch, L., "A Modal Analysis for the Response of Linear Gyroscopic Systems," *Journal of Applied Mechanics*, Vol. 42, June 1975, pp. 446-450.

[9] Meirovitch, L. and Öz, H., "Modal Control of Distributed Gyroscopic Systems," AIAA Paper 78-1421, AIAA/AAS Astrodynamics Conference, Palo Alto, Calif., Aug. 7-9, 1978.

[10] Meirovitch, L. and Öz, H., "Modal-Space Control of Large Flexible Spacecraft Possessing Ignorable Coordinates," Paper 45, Second VPI&SU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Va., June 21-23, 1979.

[11] Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill Book Co., New York, 1970.

[12] Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff and Noordhoff International Publishers, Alphen aan den Rijn, The Netherlands (to appear).

[13] Kwakernaak, H. and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.